Analytic Partial Derivatives for Estimating Low-Thrust Parameters

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Analytic partial derivatives for estimating orbital low-thrust parameters via differential correction are developed and compared with two different numerical methods. The formulation is independent of the particular thrust model used and is applicable to all physically possible elliptic orbits. The starting point for the development is the set of variational equations of the elliptic orbital elements in the form due to Lagrange. The first time derivatives of the elements are transformed to derivatives with respect to the space variable, true anomaly, and integrated to first order in closed form in a straightforward general perturbations approach, with one exception: Particular attention is given to the mean anomaly as influenced by thrust perturbations in the semimajor axis so that the complete first-order effect is included. The partials of the elements are then taken with respect to any given thrust parameter. Two comparisons are made with numerical methods for computing these thrust partials: numerical quotients and numerical integration of the variational equations for thrust. The comparisons involve a 24-hr synchronous satellite and a satellite designed for experimenting with electric propulsion systems. The results support the analytic method as an alternative to current numerical methods due to the potential saving in the initial implementation effort; and the analytic method being 70 to 98% less time consuming during execution, while still yielding the required accuracy.

Nomenclature

a = semimajor axis of elliptical two-body motion
e = eccentricity of elliptical two-body motion
i = inclination of the two-body orbital plane
ω = argument of perigee of elliptical two-body motion
Ω = longitude of the ascending node of elliptical two-body motion
M = mean anomaly of elliptical two-body motion
$\tilde{\omega}$ = $\omega + \Omega$
ε = $\omega + \Omega + M$
ε' = defined such that $\dot{\varepsilon} = \dot{\varepsilon}' + (t - t_0)\, dn/dt$
t = time
f = true anomaly
E = eccentric anomaly
n = mean angular velocity
u = $\omega + f$
r = magnitude of satellite geocentric position vector
λ = geocentric inertial latitude of satellite subpoint
μ = product of Earth mass and universal gravitation constant
ϕ = $\sin^{-1} e$, i.e., $\sin\phi = e$
S = force component acting on satellite outward along the radius vector
T = force component acting on satellite along $\mathbf{W} \times \mathbf{S}$
W = force component acting on satellite along cross product of position and velocity vectors, $\mathbf{r} \times \mathbf{v}$
p_j = model parameter associated with the satellite or its environment
$\dot{a} = da/dt$ = similarly for other time derivatives

Introduction

THE orbit of an artificial Earth satellite which is controlled by rockets of 10^{-3} to 10^{-6} lb thrust cannot generally be determined to the required accuracy[1] due to thrust uncertainties. Partial derivatives are developed here for use in a differential correction process in which the performance of the thrusters is inferred from satellite tracking observations through the effect the thrusting has on the satellite orbit. Previous methods of computing these thrust partials[2] are numerical and can be as exact as required. Numerical methods are, however, time consuming. Since the manner in which the partials are used does not require an accuracy beyond two or three places, an approximation requiring less computing time is useful. An analytic approximation to these thrust partials is presented here. It is a first-order approximation which assumes that the thrusters are fixed rigidly to the spacecraft, and that the spacecraft maintains a nearly constant orientation to both the local vertical and the orbital plane over any given sampling interval for which differential correction is being applied.

In this development the variational equations of the elliptic elements are integrated to first order in closed form, then the partials of the elements are taken with respect to any given thrust parameter. The integration in closed form is done in the same manner as in Ref. 1 except for the treatment of mean anomaly. In Ref. 1, the first-order effect of thrust on mean motion, through the effect which thrust has on the semi-major axis, is neglected. In this development this effect is included in the manner indicated in Ref. 3.

Comparisons are made with numerical solutions[4] involving a 24-hr synchronous communications satellite over two revolutions and a Space Electric Rocket Test (SERT) satellite over ten revolutions. The results strongly support the usefulness of the analytic approximations.

Analytic Thrust Partials

To determine the partial derivatives of satellite position and velocity with respect to any given thrust parameter, the variational equations of the elliptic elements are first integrated in closed form for all terms involving thrust. For all other force terms, the integration need only be indicated since all non-thrust terms drop out in the subsequent paratial differentation of the elliptic elements with respect to thrust parameters. Finally, the partials of the elliptic elements are transformed to partials of position and velocity through use of the Jacobian of the transformation from elliptic elements to the equivalent satellite position and velocity.

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Closed-Form Integration for Perturbed Motion

The variational equations of the elements are

$$\dot{a} = 2(a^3/\mu)^{1/2}[S \tan\phi \sin f + T \sec\phi(1 + e \cos f)] \quad (1)$$

$$\dot{e} = (a/\mu)^{1/2} \cos\phi[S \sin f + T(\cos f + \cos E)] \quad (2)$$

$$\dot{i} = rW \cos u/[\cos\phi(\mu a)^{1/2}] \quad (3)$$

$$\dot{\Omega} = rW \sin u/[\cos\phi \sin i(\mu a)^{1/2}] \quad (4)$$

$$\dot{\omega} = -Sa \cos \phi \cos f/[\sin \phi(\mu a)^{1/2}] + Tr \sin f(2 + \sin\phi \cos f)/[\sin\phi \cos\phi(\mu a)^{1/2}] + Wr \sin u(\tan i/2 - \csc i)/[\cos\phi(\mu a)^{1/2}] \quad (5)$$

$$\dot{M} = (a/\mu)^{1/2}(\cos^2\phi/\sin \phi)\{S[\cos f - 2 \sin\phi/(1 + e \cos f)] - T[\sin f/(1 + e \cos f) + \sin f]\} + \int (dn/dt)\, dt \quad (6)$$

The last term in the $\dot{M}$ equation replaces the usual mean motion, n, for the reasons expressed in Ref. 3, pp. 147–148.

To change the variable of integration from time to true anomaly, form the expression[5]

$$\frac{dt}{df} = \frac{[a(1 - e^2)]^{3/2}}{(\mu)^{1/2}(1 + e \cos f)^2} \quad (7)$$

and multiply both sides of each variational equation by dt/df. This is a two-body approximation to the real time variation of f. In addition to changing integration variables, all dependence on f is made explicit, and $\sin\phi$ is replaced by e for notational purposes. The transformed equations are as follows:

$$\frac{da}{df} = \frac{2a^3}{\mu}(1 - e^2)\left[S\frac{e \sin f}{(1 + e \cos f)^2} + T\frac{1}{(1 + e \cos f)}\right] \quad (8)$$

$$\frac{de}{df} = \frac{a^2}{\mu}(1 - e^2)^2\left[S\frac{\sin f}{(1 + e \cos f)^2} + T\left(\frac{\cos f}{(1 + e \cos f)^2} + \frac{\cos f}{(1 + e \cos f)^3} + \frac{e}{(1 + e \cos f)^3}\right)\right] \quad (9)$$

$$\frac{di}{df} = \frac{a^2}{\mu}(1 - e^2)^2 W\left[\cos\omega\frac{\cos f}{(1 + e \cos f)^3} - \sin\omega\frac{\sin f}{(1 + e \cos f)^3}\right] \quad (10)$$

$$\frac{d\omega}{df} = \frac{a^2}{\mu}\frac{(1 - e^2)^2}{e}\left[-S\frac{\cos f}{(1 + e \cos f)^2} + T\left[\frac{\sin f}{(1 + e \cos f)^3} + \frac{\sin f}{(1 + e \cos f)^2}\right] + We(\tan i/2 - \csc i)\left(\cos\omega\frac{\sin f}{(1 + e \cos f)^3} + \sin\omega\frac{\cos f}{(1 + e \cos f)^3}\right)\right] \quad (11)$$

$$\frac{d\Omega}{df} = \frac{a^2}{\mu}\frac{(1 - e^2)^2}{\sin i}W\left[\sin\omega\frac{\cos f}{(1 + e \cos f)^3} + \cos\omega\frac{\sin f}{(1 + e \cos f)^3}\right] \quad (12)$$

$$\frac{dM}{df} = \frac{a^2}{\mu}\frac{(1 - e^2)^{5/2}}{e}\left[S\left(\frac{\cos f}{(1 + e \cos f)^2} - \frac{2e}{(1 + e \cos f)^3}\right) - T\left(\frac{\sin f}{(1 + e \cos f)^3} + \frac{\sin f}{(1 + e \cos f)^2}\right)\right] + \int dn\frac{dt}{df} \quad (13)$$

In performing the integration from f_0 to f, six distinct integrals are involved, as given below[6,7]

$$I_1 = \int_{f_0}^{f}\frac{\sin f'\, df'}{(1 + e \cos f')^2} = \frac{\cos f_0 - \cos f}{(1 + e \cos f_0)(1 + e \cos f)} \quad (14)$$

$$I_2 = \int_{f_0}^{f}\frac{df'}{(1 + e \cos f')} = (1 - e^2)^{-1/2}(E - E_0) \quad (15)$$

$$I_3 = \int_{f_0}^{f}\frac{\sin f'\, df'}{(1 + e \cos f')^3} = \frac{e(\cos^2 f_0 - \cos^2 f) + 2(\cos f_0 - \cos f)}{2(1 + e \cos f_0)^2(1 + e \cos f)^2} \quad (16)$$

$$I_4 = \int_{f_0}^{f}\frac{\cos f'\, df'}{(1 + e \cos f')^2} = \left[\frac{\sin f'}{(1 - e^2)(1 + e \cos f')}\right]_{f_0}^{f} - e(1 - e^2)^{-1}I_2 \quad (17)$$

$$I_5 = \int_{f_0}^{f}\frac{\cos f'\, df'}{(1 + e \cos f')^3} = \frac{I_2}{e(1 - e^2)} - \frac{I_6}{e} - \left[\frac{\sin f'}{(1 - e^2)(1 + e \cos f')}\right]_{f_0}^{f} \quad (18)$$

$$I_6 = \int_{f_0}^{f}\frac{df'}{(1 + e \cos f')^3} = \frac{(2 + e^2)}{2(1 - e^2)^2}I_2 - \left[\frac{e \sin f'(4 - e^2 + 3e \cos f')}{2(1 - e^2)^2(1 + e \cos f')^2}\right]_{f_0}^{f} \quad (19)$$

Because of the importance of the term

$$\int dn\frac{dt}{df}$$

in arriving at a complete first-order approximation to the satellite motion as a function of thrust, special attention will be given to its evaluation. To arrive at a complete first-order approximation, all explicit dependence on thrust must be considered in the expressions for the variation of the elements, and all elements on the right hand side of these equations are to be considered as constants. The appearance of the differential, dn, in the $\dot{M}$ equation represents an explicit appearance of thrust, as shown below. Change from the variation of n to that of a,

$$dn = -3n\,da/2a \quad (20)$$

The instantaneous rate of change of all elements, as influenced explicitly by thrust is required. The mean anomaly rate is not only influenced by the explicit appearance of thrust but also by the direct appearance of the semimajor axis rate, which in turn has explicit thrust dependence. With the foregoing as a basis, the development proceeds as follows:

$$\frac{dM}{df} = (\text{other terms}) + \left(\int dn\right)\frac{dt}{df} = (\text{other terms}) - \frac{3}{2}\left(\int\frac{n}{a}da\right)\frac{dt}{df} \quad (21)$$

$$= (\text{other terms}) - \frac{3n}{2a}\left(\int da\right)\frac{dt}{df}$$

$$= (\text{other terms}) - \frac{3n}{2a}(a(p_j) - a_0)\frac{dt}{df} \quad (22)$$

where the explicit dependence of a on thrust is indicated by p_j which represents any thrust parameter. Noting that the integration of dM/df requires the results of integrating da/df, the complete results of integration are as follows:

$$a = a_0 + \frac{2a^3}{\mu}(1 - e^2)(S_p eI_1 + T_p I_2) + O_a \quad (23)$$

$$e = e_0 + \frac{a^2}{\mu}(1 - e^2)^2[S_p I_1 + T_p(I_4 + I_5 + eI_6)] + O_e \quad (24)$$

$$i = i_0 + \frac{a^2}{\mu}(1 - e^2)^2 W_p(I_5 \cos\omega - I_3 \sin\omega) + O_i \quad (25)$$

$$\omega = \omega_0 + \frac{a^2}{\mu}\frac{(1-e^2)^2}{e}(-S_p I_4 + T_p(I_1+I_3) + W_p e(\tan i/2 - \csc i)(I_3\cos\omega + I_5\sin\omega)) + O_\omega \quad (26)$$

$$\Omega = \Omega_0 + \frac{a^2}{\mu}\frac{(1-e^2)^2}{\sin i} W_p(I_5\sin\omega + I_3\cos\omega) + O_\Omega \quad (27)$$

$$M = M_0 + \frac{a^2}{\mu}\frac{(1-e^2)^{5/2}}{e}\left(S_p\left[I_4 - 2eI_6 + \frac{3e^2}{(1+e\cos f_0)}(I_5 - I_6\cos f_0)\right] - T_p\left\{I_1 + I_3 + \frac{3e}{(1-e^2)^{3/2}}\left[\frac{(E-E_0)^2}{2} - e[(E-E_0)\sin E + \cos E - \cos E_0]\right]\right\}\right) + O_M \quad (28)$$

Where the six definite integrals are as previously defined, S_p, T_p, and W_p denote the thrust components of the total force vector, and O_a, O_i, etc., denote the other terms in the integration resulting from nonthrust perturbation forces.

Partials of the Orbital Elements with Respect to Thrust

Noting that, $\partial O_a/\partial p_j = \partial O_e/\partial p_j = \cdots = O = \partial a_0/\partial p_j = \partial e_0/\partial p_j = \cdots$, we have for the partial derivatives of the elliptic elements with respect to a given thrust parameter, dropping the subscript from S, T, and W

$$\frac{\partial a}{\partial p_j} = \frac{2a^3}{\mu}(1-e^2)\left(eI_1\frac{\partial S}{\partial p_j} + I_2\frac{\partial T}{\partial p_j}\right) \quad (29)$$

$$\frac{\partial e}{\partial p_j} = \frac{a^2}{\mu}(1-e^2)^2\left[I_1\frac{\partial S}{\partial p_j} + (I_4 + I_5 + eI_6)\frac{\partial T}{\partial p_j}\right] \quad (30)$$

$$\frac{\partial i}{\partial p_j} = \frac{a^2}{\mu}(1-e^2)^2(I_5\cos\omega - I_3\sin\omega)\frac{\partial W}{\partial p_j} \quad (31)$$

$$\frac{\partial \omega}{\partial p_j} = \frac{a^2}{e\mu}(1-e^2)^2\left[-I_4\frac{\partial S}{\partial p_j} + (I_1 + I_3)\frac{\partial T}{\partial p_j} + e(\tan i/2 - \csc i)(I_3\cos\omega + I_5\sin\omega)\frac{\partial W}{\partial p_j}\right] \quad (32)$$

$$\frac{\partial \Omega}{\partial p_j} = \frac{a^2}{\mu}\frac{(1-e^2)^2}{\sin i}(I_5\sin\omega + I_3\cos\omega)\frac{\partial W}{\partial p_j} \quad (33)$$

$$\frac{\partial M}{\partial p_j} = \frac{a^2}{\mu e}(1-e^2)^{5/2}\left(\left[I_4 - 2eI_6 + \frac{3e^2}{1+e\cos f_0}(I_5 - I_6\cos f_0)\right]\frac{\partial S}{\partial p_j} - \left\{I_1 + I_3 + \frac{3e}{(1-e^2)^{3/2}}\left[\frac{(E-E_0)^2}{2} - e[(E-E_0)\sin E + \cos E - \cos E_0]\right]\right\}\frac{\partial T}{\partial p_j}\right) \quad (34)$$

These six expressions represent a first order approximation to the thrust partials at any time measured from an epoch at which they are all zero by definition.

Transformation from Element Partials to Satellite Position and Velocity Partials

The transformation, denoted as

$$\frac{\partial(X,Y,Z,\dot X,\dot Y,\dot Z)}{\partial p_j} = \frac{\partial(X,Y,Z,\dot X,\dot Y,\dot Z)}{\partial(a,e,i,\omega,\Omega,M)}\frac{\partial(a,e,i,\omega,\Omega,M)}{\partial p_j} \quad (35)$$

derives directly from taking the partial derivatives of satellite position and velocity, expressed as functions of the elements, with respect to the elements. The resulting Jacobian necessary for the transformation is given in Table 1 (Ref. 8).

where

$$A = -(e + \cos E)/(1-e^2) \quad (36)$$

$$B = (a^3/\mu)^{1/2}\sin E[eA + 2/(1-e^2)] \quad (37)$$

$$C = (\mu/a)^{1/2}\frac{\sin E}{(1-e)^2 r^3}[r^2 - ar - a^2(1-e^2)] \quad (38)$$

$$D = \cos E/(1-e^2) \quad (39)$$

$$G = (a^3/\mu)^{1/2} \quad (40)$$

$$H = -(a^3\mu)^{1/2}/r^3 \quad (41)$$

Comparison with Numerical Solution

The equations for the thrust partials were programed and tested against two numerical solutions. The first test case was two revolutions of a synchronous satellite of the Applications Technology Satellite (ATS) type. The particular thrust parameter involved was thrust acceleration. The numerical

Table 1 $\partial(X,Y,Z,\dot X,\dot Y,\dot Z)/\partial(a,e,i,\omega,\Omega,M)$

	X	Y	Z	$\dot X$	$\dot Y$	$\dot Z$
$\frac{\partial}{\partial a}$	X/a	Y/a	Z/a	$-\dot X/2a$	$-\dot Y/2a$	$-\dot Z/2a$
$\frac{\partial}{\partial e}$	$AX + B\dot X$	$AY + B\dot Y$	$AZ + B\dot Z$	$CX + D\dot X$	$CY + D\dot Y$	$CZ + D\dot Z$
$\frac{\partial}{\partial i}$	$Z\sin\Omega$	$-Z\cos\Omega$	$-X\sin\Omega + Y\cos\Omega$	$\dot Z\sin\Omega$	$-\dot Z\cos\Omega$	$-\dot X\sin\Omega + \dot Y\cos\Omega$
$\frac{\partial}{\partial \omega}$	$-Z\sin i\cos\Omega - Y\cos i$	$-Z\sin i\sin\Omega + X\cos i$	$\sin i(X\cos\Omega + Y\sin\Omega)$	$-\dot Z\sin i\cos\Omega - \dot Y\cos i$	$-\dot Z\sin i\sin\Omega + \dot X\cos i$	$\sin i(\dot X\cos\Omega + \dot Y\sin\Omega)$
$\frac{\partial}{\partial \Omega}$	$-Y$	X	0	$-\dot Y$	$\dot X$	0
$\frac{\partial}{\partial M}$	$G\dot X$	$G\dot Y$	$G\dot Z$	HX	HY	HZ

solution was arrived at through the approximations for the ith thrust partial

$$\partial \mathbf{r}_i/\partial T \cong [\mathbf{r}(T+\Delta T, t_i) - \mathbf{r}(T, t_i)]/\Delta T \tag{42}$$

$$\partial \dot{\mathbf{r}}_i/\partial T \cong [\dot{\mathbf{r}}(T+\Delta T, t_i) - \dot{\mathbf{r}}(T, t_i)]/\Delta T \tag{43}$$

where T is thrust acceleration, and ΔT is an increment of thrust acceleration.

The accuracy of these quotients is sensitive to the particular value of ΔT being used. Several different values of ΔT were tested so as to establish a convergent pattern. The optimum value was then chosen for computing the partials. This test case involved the partials of radial distance, longitude, and their time derivatives instead of the six position and velocity components.[4] The conversions necessary for comparison to Ref. 4 are

$$\partial r/\partial T = (\partial \mathbf{r}/\partial T)\cdot(\mathbf{r}/r) \tag{44}$$

$$\partial \lambda/\partial T = [\mathbf{r}x(\partial \mathbf{r}/\partial T)]_z/(X^2 + Y^2) \tag{45}$$

$$\partial \dot{r}/\partial T = (\dot{\mathbf{r}}/r)\cdot[(\partial \mathbf{r}/\partial T) - (\mathbf{r}/r)(\partial r/\partial T)] + (\mathbf{r}/r)\cdot(\partial \dot{\mathbf{r}}/\partial T) \tag{46}$$

$$\partial \dot{\lambda}/\partial T = \{[\mathbf{r}x(\partial \dot{\mathbf{r}}/\partial T)]_z - [\dot{\mathbf{r}}x(\partial \mathbf{r}/\partial T)]_z - 2\dot{\lambda}[X(\partial X/\partial T) + Y(\partial Y/\partial T)]\}/(X^2 + Y^2) \tag{47}$$

where λ is longitude, $(\,)_z$ denotes the z component of the vector in the parenthesis, and

$$\mathbf{r} \equiv (X, Y, Z) \tag{48}$$

$$r \equiv |\mathbf{r}| \tag{49}$$

$$\dot{r} = (\mathbf{r}/r)\cdot\dot{\mathbf{r}} \tag{50}$$

$$\lambda = \tan^{-1}(Y/X) \tag{51}$$

$$\dot{\lambda} = (X\dot{Y} - Y\dot{X})/(X^2 + Y^2) \tag{52}$$

Graphs of the percentage deviation of the analytic from the numerical for $\partial\lambda/\partial T$ and $\partial\dot{r}/\partial T$ are given in Fig. 1.

The second test case was ten revolutions of the SERT II satellite using thrust magnitude as a parameter. The SERT II values were computed using the Definitive Orbit Determination System of the Mission and Trajectory Analysis Division of the Goddard Space Flight Center, NASA. The technique employed was numerical integration of the complete variational equations of the osculating elements with respect to thrust magnitude. No conversion of the partials for comparison was necessary. The results are given in Table 2. Note that the differences are given in terms of percentage differences.

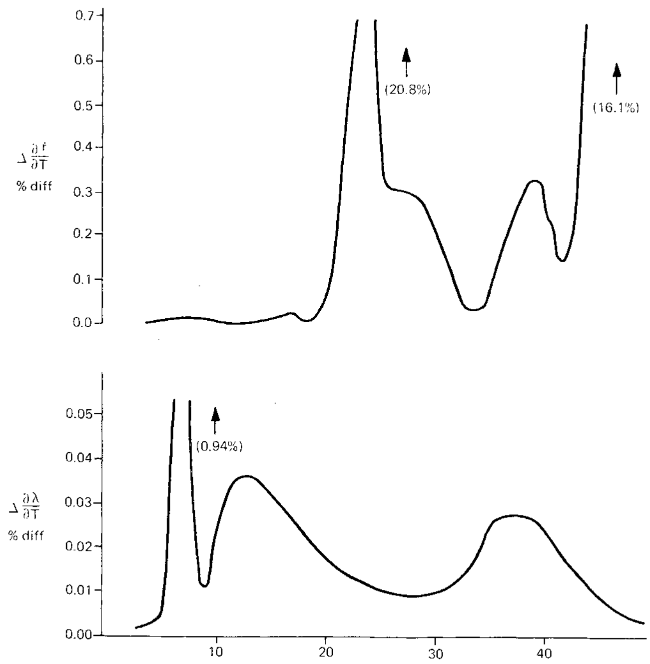

Fig. 1 ATS test case, $\Delta(\partial\lambda/\partial T)$, $\Delta(\partial\dot{r}/\partial T)$ numerical vs analytical.

Table 2 SERT II test case: numerical vs analytic

	$\partial X/\partial T$	$\partial Y/\partial T$	$\partial Z/\partial T$	$\partial \dot{X}/\partial T$	$\partial \dot{Y}/\partial T$	$\partial \dot{Z}/\partial T$
Time = 1 hr						
SERT II	0.16522-5	0.2149-5	– 0.1068-5	– 0.27369-6	0.2100	0.6877
Analytic	0.16524-5	0.2137-5	– 0.1092-5	– 0.27369-6	0.2094	0.7099
% Diff.	0.01	0.50	2.4	0.0	0.33	2.9
Time = 4 hr						
SERT II	– 0.8412-6	– 0.4446-4	– 0.1989-4	– 0.5259-5	0.1223-4	– 0.4018-4
Analytic	– 0.8467-6	– 0.4470-4	– 0.1979-4	– 0.5261-5	0.1222-4	– 0.4004-4
% Diff.	0.60	0.68	0.51	0.06	0.08	0.26
Time = 7 hr						
SERT II	0.2202-4	– 0.1790-4	0.1494-3	0.2155-5	– 0.1310-3	– 0.4894-5
Analytic	0.2211-4	– 0.1795-4	0.1497-3	0.2151-5	– 0.1312-3	– 0.4874-5
% Diff.	0.45	0.28	0.20	0.19	0.15	0.41
Time = 10 hr						
SERT II	– 0.1259-4	0.3028-3	– 0.4487-4	0.3448-4	0.5508-4	0.2538-3
Analytic	– 0.1250-4	0.3012-3	– 0.4467-4	0.3457-4	0.5500-4	0.2530-3
% Diff.	0.72	0.53	0.45	0.29	0.14	0.32
Time = 13 hr						
SERT II	– 0.6095-4	– 0.2073-3	– 0.4727-3	– 0.3759-4	0.4022-3	– 0.1902-3
Analytic	– 0.6115-4	– 0.2076-3	– 0.4734-3	– 0.3768-4	0.4028-3	– 0.1900-3
% Diff.	0.3	0.14	0.15	0.27	0.15	0.11
Time = 16 hr						
SERT II	0.8841-4	– 0.6188-3	0.4896-3	– 0.6116-4	– 0.4467-3	– 0.5166-3
Analytic	0.8832-4	– 0.6176-3	0.4886-3	– 0.6123-4	0.4468-3	– 0.5154-3
% Diff.	0.11	0.16	0.21	0.11	0.02	0.19

Discussion of Results

Taking each of the four components in the first test case individually, we have the following results of the comparison described previously:

Neglecting the values near the 24 and 48 hr points, $\Delta(\partial\dot{r}/\partial T)$ was less than 0.4%. These values would have a negligible effect on the solution for thrust parameters, anyway, since the partials for all observation points in the orbit are summed together, and the values of $\partial\dot{r}/\partial T$ near the 24 and 48 hr points are relatively very small.

The SERT II test case results in Table 2 show that the relative difference, component for component, is less than 0.9% in 90 out of the 96 values involved. Of the remaining six, five are less than 3% and the sixth is 10.4%. In this case, the 10.4% value was for the smallest component by two orders of magnitude. The mean difference over all values is 0.47%, and neglecting the 10.4% value (which is valid since the absolute difference is much less), the mean difference becomes 0.36%.

Conclusions

Using this analytic method for computing thrust partials, a mean accuracy of 0.5% is attainable over at least two revolutions of a synchronous satellite and ten revolutions of a SERT type satellite. Because of the fact that the orbit improvement procedure is iterative, and that these partial derivatives represent tangents to hyper-surfaces (to a curve when only one parameter is being estimated), mean accuracies of from one to five percent are entirely adequate. It has been the author's experience that some points being off by as much as 50% have not adversely affected the overall solution. Therefore, as presented in this paper, it is likely that the analytic approximations to the partial derivatives of instantaneous satellite position and velocity with respect to thrust parameters over a time span of at least two days, could replace the corresponding numerical partials for use with synchronous communciations type satellites and SERT type satellites, having gravity gradient attitude control.

As for general application, it could be inferred from the error graphs that up to a seven day time span of the synchronous satellites could apply. Further testing might be required for the SERT case. Note that after ten orbits the analytic partials show no sign of getting worse. Further, regarding general application, there are no restraints on the types of physical orbits this formulation can be used for. The singularities involving eccentricity and inclination in the equations for $\partial\omega/\partial p_j$ and $\partial\Omega/\partial p_j$ are extremely unlikely to occur in real satellite orbits. Zero eccentricity and inclination are very difficult to achieve. The first test case used here was not real. It had $e \cong 10^{-4}$ and $i \cong 10^{-15}$ rad. No computational difficulty was encountered. Large-scale computers typically used in orbital work can handle values $\sim 10^{-78}$. The important point is that $\partial\omega/\partial p_j$ and $\partial\Omega/\partial p_j$ approach discontinuity at $e = i = 0$ in a smooth manner. If for some reason zero values for e and i were encountered, a simple program check could accommodate them.

The motivation for using these analytic thrust partials is the relative ease of implementation and the 70% to 98% saving in computer time over any numerical method of comparable accuracy.†

Table 3 ATS test case, deviations of analytical from numerical

Actual deviations	Mean deviations
$0 < \Delta \dfrac{\partial r}{\partial T} < 0.05\%$	$\Delta \dfrac{\partial r}{\partial T} \simeq 0.006\%$
$0 < \Delta \dfrac{\partial \lambda}{\partial T} < 1.0\%$	$\Delta \dfrac{\partial \lambda}{\partial T} \simeq 0.05\%$
$0 < \Delta \dfrac{\partial \dot{r}}{\partial T} < 21.0\%$	$\Delta \dfrac{\partial \dot{r}}{\partial T} \simeq 0.2\%$
$0 < \Delta \dfrac{\partial \dot{\lambda}}{\partial T} < 0.08\%$	$\Delta \dfrac{\partial \dot{\lambda}}{\partial T} \simeq 0.01\%$

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† The author's own estimate based on experience in implementing and using both kinds.